

Decomposing various graphs into short even-length cycles

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Abstract

We prove that a complete bipartite graph can be decomposed into cycles of arbitrary specified lengths provided that the obvious necessary conditions are satisfied, the length of each cycle is at most the size of the smallest part, and the longest cycle is at most three times as long as the second longest. We then use this result to obtain results on incomplete even cycle systems with a hole and on decompositions of complete multipartite graphs into cycles of uniform even length.

Keywords: graph decomposition; cycle; complete bipartite graph; incomplete cycle system; complete multipartite graph

1 Introduction

A decomposition of a graph G is a set of subgraphs of G whose edge sets partition the edge set of G . In 1981 Alspach [1] posed the problem of determining when a complete graph can be decomposed into cycles of arbitrary specified lengths. While a complete

solution to this problem continues to be elusive, a great deal of work has been done on it (see the survey [5], for example). Less attention has been paid to the problem of determining when a complete bipartite graph can be decomposed into cycles of arbitrary specified lengths. Sotteau [20] has solved the problem in the case of cycles of uniform length. Chou, Fu and Huang [13] have given a solution in the case when all the cycles have lengths in $\{4, 6, 8\}$. Chou and Fu [12] have also examined the problem in the case when all the cycles have lengths in $\{4, m\}$, for some even integer m , and have given a complete solution for $m = 10$ and $m = 12$. Decompositions of complete bipartite graphs with equal sized parts into cycles and a perfect matching have also been studied, with the case of uniform length cycles largely solved by Archdeacon et al. [3] and the solution completed by Ma, Pu and Shen [17]. In [13] and [12] the analogues of the results mentioned above for decompositions into cycles and a perfect matching are also obtained.

Let $K_{a,b}$ denote the complete bipartite graph with parts of size a and b , and let $K_{a,a} - I$ denote the graph obtained from $K_{a,a}$ by removing the edges of a perfect matching. The main results of this paper are as follows.

Theorem 1.1 *Let a and b be positive integers such that either a and b are even or $a = b$, and let $K_{a,b}^*$ be the graph $K_{a,b}$ if a and b are even and the graph $K_{a,b} - I$ if $a = b$ and a is odd. If m_1, m_2, \dots, m_t are even integers such that $4 \leq m_1 \leq m_2 \leq \dots \leq m_t \leq \min(a, b, 3m_{t-1})$ and $m_1 + m_2 + \dots + m_t = |E(K_{a,b}^*)|$, then there is a decomposition of $K_{a,b}^*$ into cycles of lengths m_1, m_2, \dots, m_t .*

Note that the conditions of Theorem 1.1 are necessary for the decomposition to exist, with the exception of requiring that $m_t \leq \min(a, b, 3m_{t-1})$. It is necessary that $m_t \leq 2 \min(a, b)$, however.

We shall apply Theorem 1.1 to obtain a result concerning even cycle systems with a hole and a result concerning decompositions of complete multipartite graphs into uniform even-length cycles.

For positive integers u and v with $u \leq v$, let $K_v - K_u$ denote the graph obtained from the complete graph of order v by removing the edges of a complete subgraph of order u . A decomposition of $K_v - K_u$ into m -cycles is often called an *incomplete m -cycle system of order v with a hole of size u* . Incomplete cycle systems have received considerable attention (see [9] and [14], for example). We will concern ourselves here only with incomplete even cycle systems. Bryant, Rodger and Spicer [10] have found necessary and sufficient conditions for the existence of an incomplete m -cycle system of order v with a hole of size u for all even m with $m \leq 14$, and also when $v \equiv u \pmod{2m}$. We prove the following result.

Theorem 1.2 *Let v and u be odd positive integers. If $m \geq 4$ is an even integer such that $\binom{v}{2} - \binom{u}{2} \equiv 0 \pmod{m}$, $u \geq m + 1$, and $v - u \geq m$, then there is an m -cycle decomposition of $K_v - K_u$.*

Note that the condition of the theorem that $\binom{v}{2} - \binom{u}{2} \equiv 0 \pmod{m}$ is clearly necessary, and also that if we demand that u be the order of a non-trivial m -cycle system (so we can “fill the hole” and obtain an m -cycle system of order v with a subsystem of size u), then the condition that $u \geq m + 1$ is also necessary.

Decompositions of complete multipartite graphs into cycles have also been well studied (see, for example, the survey [4] and the references therein). Again, we will only consider decompositions into cycles of uniform even length. Laskar and Auerbach [16] have shown that a complete multipartite graph with all parts of equal size has a decomposition into hamilton cycles if its vertices have even degree, and has a decomposition into hamilton cycles and a perfect matching if its vertices have odd degree. Cavenagh and Billington [11] have established necessary and sufficient conditions for the existence of a decomposition of a complete multipartite graph into m -cycles for each $m \in \{4, 6, 8\}$. Our result is as follows.

Theorem 1.3 *Let G be a complete multipartite graph with parts of even size. If $m \geq 4$ is an even integer such that every part of G has size at least $m + 2$ and $|E(G)| \equiv 0 \pmod{m}$, then there is a decomposition of G into m -cycles.*

2 Preliminary Results

Our main goal in Sections 2, 3 and 4 is to prove Theorem 1.1. In this section our aim is to prove Lemmas 2.1, 2.2 and 2.4, which we will require in Section 3. Lemma 2.3 is used only in the proof of Lemma 2.4.

We require some more notation before we proceed. The *size* of a graph is the number of edges it contains. We say that a graph is *even* if each of its vertices has even degree and that it is *odd* if each of its vertices has odd degree. The *neighbourhood* of a vertex x in a graph G , denoted $\text{Nbd}_G(x)$, is the set of all the vertices adjacent in G to x (not including x itself). If $M = m_1, m_2, \dots, m_t$ is a list of integers and G is a graph which is either even or odd, then an $(M)^*$ -*packing* of G is a decomposition of some subgraph of G into t cycles of lengths m_1, m_2, \dots, m_t if G is an even graph, and into t cycles of lengths m_1, m_2, \dots, m_t and a perfect matching if G is an odd graph. The *leave* of an $(M)^*$ -packing of G is the complement in G of the subgraph of G which is decomposed. In other words, the leave is the spanning subgraph of G whose edges are precisely those which do not appear in the cycles or the perfect matching (if it exists) of the packing. Note that the leave of an $(M)^*$ -packing is always an even graph. An $(M)^*$ -decomposition of G is an $(M)^*$ -packing of G whose leave is an empty graph. If we know G to be an even graph, then we will drop the asterisk and simply refer to an (M) -packing or (M) -decomposition of G .

The *length* of a cycle or path is the number of edges it contains. A cycle of length p is called a p -cycle and a path of length q is called a q -path. The p -cycle with vertices x_1, x_2, \dots, x_p and edges $x_1x_2, x_2x_3, \dots, x_{p-1}x_p, x_px_1$ is denoted by (x_1, x_2, \dots, x_p) and the q -path with vertices y_0, y_1, \dots, y_q and edges $y_0y_1, y_1y_2, \dots, y_{q-1}y_q$ is denoted by $[y_0, y_1, \dots, y_q]$. We will say that y_0 to y_q are the *end vertices* of such a path, and that the path is from y_0 to y_q (or from y_q to y_0). For graphs G and H , the union of G and H , denoted by $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

We define two special kinds of graphs which we call rings and chains. Note that

these definitions vary slightly from the definitions given in [6] to suit our purposes in this paper.

A graph G is an (a_1, a_2, \dots, a_s) -*chain* if G is the edge-disjoint union of $s \geq 2$ cycles A_1, A_2, \dots, A_s such that

- A_i is a cycle of length a_i for $i = 1, 2, \dots, s$; and
- for $1 \leq i < j \leq s$, $|V(A_i) \cap V(A_j)| = 1$ if $j = i + 1$ and $|V(A_i) \cap V(A_j)| = 0$ otherwise.

We call A_1 and A_s the *end cycles* of G . A graph is an s -*chain*, or just a *chain*, if it is an (a_1, a_2, \dots, a_s) -chain for some integers a_1, a_2, \dots, a_s . We denote a (p, q) -chain with cycles $(x_1, x_2, \dots, x_{p-1}, c)$ and $(y_1, y_2, \dots, y_{q-1}, c)$ by $(x_1, x_2, \dots, x_{p-1}, c) \cdot (c, y_1, y_2, \dots, y_{q-1})$.

A graph G is an (a_1, a_2, \dots, a_s) -*ring* if G is the edge-disjoint union of $s \geq 2$ cycles A_1, A_2, \dots, A_s such that

- A_i is a cycle of length a_i for $i = 1, 2, \dots, s$;
- if $s \geq 3$, then for $1 \leq i < j \leq s$, $|V(A_i) \cap V(A_j)| = 1$ if $j = i + 1$ or if $(i, j) = (1, s)$, and $|V(A_i) \cap V(A_j)| = 0$ otherwise; and
- if $s = 2$, then $|V(A_1) \cap V(A_2)| = 2$.

We call A_1, A_2, \dots, A_s the *ring cycles* of G (note that G contains cycles which are not ring cycles). A graph is an s -*ring*, or just a *ring*, if it is an (a_1, a_2, \dots, a_s) -ring for some integers a_1, a_2, \dots, a_s .

The techniques used in this paper are often the bipartite analogues of techniques used for complete graphs in [6] and [7]. In particular, the crucial Lemma 3.6 of this paper is the bipartite analogue of Lemma 3.1 of [7]. Unfortunately however, we have thus far been unable to find a good bipartite analogue for Lemma 1.2 of [6]. We begin by using Lemma 2.1 of [6], which deals with modifying packings of complete graphs,

to obtain a similar result for decompositions of general graphs. Given a permutation π of a set V and a graph G with $V(G) \subseteq V$, the graph $\pi(G)$ is defined to be the graph with vertex set $\{\pi(x) : x \in V(G)\}$ and edge set $\{\pi(x)\pi(y) : xy \in E(G)\}$.

Lemma 2.1 *Let G be a graph, let M be a list of integers, let \mathcal{P} be an $(M)^*$ -packing of G , let L be the leave of \mathcal{P} , and let α and β be vertices in G such that $\text{Nbd}_G(\alpha) = \text{Nbd}_G(\beta)$, and let π be the transposition $(\alpha\beta)$. Then there exists a partition of the set $(\text{Nbd}_L(\alpha) \cup \text{Nbd}_L(\beta)) \setminus ((\text{Nbd}_L(\alpha) \cap \text{Nbd}_L(\beta)) \cup \{\alpha, \beta\})$ into pairs such that for each pair $\{u, v\}$ of the partition, there exists an $(M)^*$ -packing \mathcal{P}' of G whose leave L' differs from L only in that αu , αv , βu and βv are edges in L' if and only if they are not edges in L . Furthermore, if $\mathcal{P} = \{C_1, C_2, \dots, C_t\}$ (G even) or $\mathcal{P} = \{F, C_1, C_2, \dots, C_t\}$ (G odd) where C_1, C_2, \dots, C_t are cycles and F is a perfect matching, then $\mathcal{P}' = \{C'_1, C'_2, \dots, C'_t\}$ (G even) or $\mathcal{P}' = \{F', C'_1, C'_2, \dots, C'_t\}$ (G odd), where, for $i = 1, 2, \dots, t$, C'_i is a cycle of the same length as C_i and F' is a perfect matching such that*

- (i) *either $F' = F$ or $F' = \pi(F)$;*
- (ii) *for $i = 1, 2, \dots, t$ if neither α nor β is in $V(C_i)$, then $C'_i = C_i$;*
- (iii) *for $i = 1, 2, \dots, t$ if exactly one of α and β is in $V(C_i)$, then either $C'_i = C_i$ or $C'_i = \pi(C_i)$; and*
- (iv) *for $i = 1, 2, \dots, t$ if both α and β are in $V(C_i)$, then $C'_i \in \{C_i, \pi(C_i), \pi(P_i) \cup P_i^\dagger, P_i \cup \pi(P_i^\dagger)\}$, where P_i and P_i^\dagger are the two paths in C_i which have end vertices α and β .*

Proof If G is an even graph and $|V(G)|$ is even let G^\dagger be the graph obtained from G by adding an isolated vertex. Otherwise, let $G^\dagger = G$. Note that we can consider \mathcal{P} as an $(M)^*$ -packing of the complete graph on vertex set $V(G^\dagger)$. The leave H of this packing is the edge-disjoint union of L and the complement of G^\dagger . By applying Lemma 2.1 of [6] we can obtain a partition of the set $(\text{Nbd}_H(\alpha) \cup \text{Nbd}_H(\beta)) \setminus ((\text{Nbd}_H(\alpha) \cap \text{Nbd}_H(\beta)) \cup \{\alpha, \beta\})$ into pairs such that for each pair $\{u, v\}$ of the partition, there

exists an $(M)^*$ -packing \mathcal{P}' of the complete graph on the vertex set $V(G^\dagger)$ whose leave H' differs from H only in that $\alpha u, \alpha v, \beta u$ and βv are edges in H if and only if they are not edges in H . Furthermore, if $\mathcal{P} = \{C_1, C_2, \dots, C_t\}$ ($|V(G^\dagger)|$ even) or $\mathcal{P} = \{F, C_1, C_2, \dots, C_t\}$ ($|V(G^\dagger)|$ odd) where C_1, C_2, \dots, C_t are cycles and F is a perfect matching, then $\mathcal{P}' = \{C'_1, C'_2, \dots, C'_t\}$ ($|V(G^\dagger)|$ even) or $\mathcal{P}' = \{F', C'_1, C'_2, \dots, C'_t\}$ ($|V(G^\dagger)|$ odd) where for $i = 1, 2, \dots, t$, C'_i is a cycle of the same length as C_i and F' is a perfect matching such that (i), (ii), (iii) and (iv) hold. Since $\text{Nbd}_G(\alpha) = \text{Nbd}_G(\beta)$ and H is the edge-disjoint union of L and the complement of G^\dagger , we have that $(\text{Nbd}_H(\alpha) \cup \text{Nbd}_H(\beta)) \setminus ((\text{Nbd}_H(\alpha) \cap \text{Nbd}_H(\beta)) \cup \{\alpha, \beta\}) = (\text{Nbd}_L(\alpha) \cup \text{Nbd}_L(\beta)) \setminus ((\text{Nbd}_L(\alpha) \cap \text{Nbd}_L(\beta)) \cup \{\alpha, \beta\})$. Furthermore, $\alpha u, \alpha v, \beta u, \beta v \in E(G)$, so H' is the edge-disjoint union of the complement of G^\dagger and a graph L' on vertex set $V(G)$ which differs from L only in that $\alpha u, \alpha v, \beta u$ and βv are edges in L' if and only if they are not edges in L . Thus, by viewing \mathcal{P}' as a packing of G , we have our result. \square

Remark. In this paper we will always use the above result in the case where G is a complete bipartite graph. Thus the condition $\text{Nbd}_G(\alpha) = \text{Nbd}_G(\beta)$ will hold if and only if α and β are in the same part of G .

When applying Lemma 2.1, we will say that \mathcal{P}' is the $(M)^*$ -packing of $K_{a,b}$ obtained from \mathcal{P} by performing the (α, β) -switch with origin u and terminus v (we could equally call v the origin and u the terminus).

In the remainder of this paper we will often prove results relating to an $(M)^*$ -packing of $K_{a,b}$, where we take it as read that M is a list of positive even integers and that a and b are positive integers such that either both a and b are even or $a = b$. Note that when a and b are even this is a packing of $K_{a,b}$ with cycles of lengths M and when $a = b$ is odd this is a packing of $K_{a,a}$ with cycles of lengths M and a perfect matching.

Lemma 2.2 *Suppose that there exists an $(M)^*$ -packing \mathcal{P} of $K_{a,b}$ with a leave of size l whose only non-trivial component H contains a path $P = [x_0, x_1, \dots, x_t]$ of even length at least 4 such that the edges in $E(H) \setminus E(P)$ form a path and such that $x_1 x_t \notin E(H)$.*

Let S be the (x_0, x_t) -switch with origin x_1 (note that x_0 and x_t are in the same part of $K_{a,b}$ since P is a path of even length) and let \mathcal{P}' be the $(M)^*$ -packing of $K_{a,b}$ obtained from \mathcal{P} by performing S . If S does not have terminus x_{t-1} , then the leave of \mathcal{P}' has a decomposition into a t -cycle and an $(l-t)$ -cycle, and there are at least as many vertices of degree 4 in the leave of \mathcal{P}' as there are in the leave of \mathcal{P} .

Remark. In particular, notice that if H contains a vertex of degree at least 4, then the leave of \mathcal{P}' will have only one non-trivial component.

Proof Since each vertex of H has even degree it is clear that $\deg_H(x_0) = \deg_H(x_t) = 2$. Let the path with edge set $E(H) \setminus E(P)$ be $[x_0 = y_0, y_1, y_2, \dots, y_{l-t} = x_t]$. Since S does not have terminus x_{t-1} , it must have terminus y_1 or y_{l-t-1} . If S has terminus y_1 , then the leave of \mathcal{P}' has a decomposition into the t -cycle (x_1, x_2, \dots, x_t) and the $(l-t)$ -cycle $(y_1, y_2, \dots, y_{l-t})$, and there is one more vertex of degree 4 in the leave of \mathcal{P}' than there is in the leave of \mathcal{P} . Otherwise S has terminus y_{l-t-1} , the leave of \mathcal{P}' has a decomposition into the t -cycle (x_1, x_2, \dots, x_t) and the $(l-t)$ -cycle $(y_0, y_1, y_2, \dots, y_{l-t-1})$, and there are the same number of vertices of degree 4 in the leave of \mathcal{P}' as there are in the leave of \mathcal{P} . \square

Lemma 2.3 *Suppose there exists an $(M)^*$ -packing of $K_{a,b}$ with a leave whose only non-trivial component is a (p, q) -chain. If m is an even integer such that $p \leq m$ and $p + q - m \geq 4$, then there exist*

- (i) *an $(M)^*$ -packing of $K_{a,b}$ with a leave whose only non-trivial component either has a decomposition into an m -cycle and a $(p + q - m)$ -cycle, or is an $(m - p + 2, 2p + q - m - 2)$ -chain; and*
- (ii) *an $(M)^*$ -packing of $K_{a,b}$ with a leave whose only non-trivial component either has a decomposition into an m -cycle and a $(p + q - m)$ -cycle, or is an $(m - p + 4, 2p + q - m - 4)$ -chain.*

Proof If $p = m$ then we are finished, so assume $p \leq m-2$. Let \mathcal{P} be an $(M)^*$ -packing of $K_{a,b}$ and let $H = (x_1, x_2, \dots, x_{p-1}, c) \cdot (c, y_1, y_2, \dots, y_{q-1})$ be the only non-trivial component in its leave.

(i) Let \mathcal{P}' be the $(M)^*$ -packing of $K_{a,b}$ obtained from \mathcal{P} by performing the (x_1, y_{m-p+1}) -switch S' with origin x_2 (note that x_1 and y_{m-p+1} are in the same part of $K_{a,b}$ since $[x_1, c, y_1, y_2, \dots, y_{m-p+1}]$ is an $(m-p+2)$ -path and $m-p+2$ is even). If S' has terminus y_{m-p} , then the only non-trivial component in the leave of \mathcal{P}' is

$$(y_1, y_2, \dots, y_{m-p}, x_1, c) \cdot (c, x_{p-1}, x_{p-2}, \dots, x_2, y_{m-p+1}, y_{m-p+2}, \dots, y_{q-1}),$$

which is an $(m-p+2, 2p+q-m-2)$ -chain. Otherwise S' does not have terminus y_{m-p} and, by Lemma 2.2 with $P = [x_1, x_2, \dots, x_{p-1}, c, y_1, y_2, \dots, y_{m-p+1}]$, the leave of \mathcal{P}' has a decomposition into an m -cycle and a $(p+q-m)$ -cycle.

(ii) Let \mathcal{P}'' be the $(M)^*$ -packing of $K_{a,b}$ obtained from \mathcal{P} by performing the (x_2, y_{m-p+2}) -switch S'' with origin x_3 (note that x_2 and y_{m-p+2} are in the same part of $K_{a,b}$ since $[x_2, x_1, c, y_1, y_2, \dots, y_{m-p+2}]$ is an $(m-p+4)$ -path and $m-p+4$ is even). If S'' has terminus y_{m-p+1} , then the only non-trivial component in the leave of \mathcal{P}'' is

$$(y_1, y_2, \dots, y_{m-p+1}, x_2, x_1, c) \cdot (c, x_{p-1}, x_{p-2}, \dots, x_3, y_{m-p+2}, y_{m-p+3}, \dots, y_{q-1}),$$

which is an $(m-p+4, 2p+q-m-4)$ -chain. Otherwise, S'' does not have terminus y_{m-p+1} and, by Lemma 2.2 with $P = [x_2, x_3, \dots, x_{p-1}, c, y_1, y_2, \dots, y_{m-p+2}]$, the leave of \mathcal{P}'' has a decomposition into an m -cycle and a $(p+q-m)$ -cycle. \square

Lemma 2.4 *Suppose there exists an $(M)^*$ -packing of $K_{a,b}$ with a leave whose only non-trivial component is a (p, q) -chain. If m_1 and m_2 are even integers such that $m_1, m_2 \geq 4$ and $m_1 + m_2 = p + q$, then there exists an $(M, m_1, m_2)^*$ -decomposition of $K_{a,b}$.*

Proof We can assume without loss of generality that $p \leq q$ and $m_1 \leq m_2$. Since $m_1 + m_2 = p + q$, this implies that $p \leq m_2$. Let $m = m_2$. We wish to find an $(M, m, p+q-m)^*$ -decomposition of $K_{a,b}$.

If $p = m$ then we are finished immediately, so assume $m > p$. We apply Lemma 2.3 (i) if $p \geq \frac{m+4}{2}$ and we apply Lemma 2.3 (ii) if $p \leq \frac{m+2}{2}$. Thus, we either obtain the required $(M)^*$ -packing of $K_{a,b}$ or we obtain an $(M)^*$ -packing of $K_{a,b}$ with a leave whose only non-trivial component is an $(m-p+2, 2p+q-m-2)$ -chain when $p \geq \frac{m+4}{2}$, and is an $(m-p+4, 2p+q-m-4)$ -chain when $p \leq \frac{m+2}{2}$. Let $p_2 = m-p+2$ if $p \geq \frac{m+4}{2}$ and let $p_2 = m-p+4$ if $p \leq \frac{m+2}{2}$. If $p_2 = m$ we are finished. Otherwise, we now apply Lemma 2.3 (i) if $p_2 \geq \frac{m+4}{2}$ and we apply Lemma 2.3 (ii) if $p_2 \leq \frac{m+2}{2}$. We claim that by repeating this process, in each instance applying Lemma 2.3 (i) when $p_i \geq \frac{m+4}{2}$, applying Lemma 2.3 (ii) when $p_i \leq \frac{m+2}{2}$, and defining p_{i+1} by $p_{i+1} = m - p_i + 2$ if $p_i \geq \frac{m+4}{2}$ and $p_{i+1} = m - p_i + 4$ if $p_i \leq \frac{m+2}{2}$, we eventually obtain the required $(M)^*$ -packing of $K_{a,b}$. To see this, observe that if $m \equiv 0 \pmod{4}$ and $p = \frac{m+4}{2}$, then the sequence p, p_2, p_3, \dots is

$$\frac{m+4}{2}, \frac{m}{2}, \frac{m+8}{2}, \frac{m-4}{2}, \dots, m-4, 6, m-2, 4, m$$

and that if $m \equiv 2 \pmod{4}$ and $p = \frac{m+2}{2}$, then the sequence p, p_2, p_3, \dots is

$$\frac{m+2}{2}, \frac{m+6}{2}, \frac{m-2}{2}, \frac{m+10}{2}, \dots, m-4, 6, m-2, 4, m.$$

In either case, the sequence contains every even integer x in the range $4 \leq x \leq m$. \square

3 Merging Cycles

Our main goal in this section is to prove Lemma 3.6 which allows us to merge a number of cycles in a decomposition into one large cycle, given the existence of a “catalyst cycle” of appropriate length. We require a number of preliminary lemmas first, however, many of which will also be used in Sections 5 and 6.

Lemma 3.1 *Suppose there exists an $(M)^*$ -packing of $K_{a,b}$, where $a \leq b$, with a leave of size l , where $l \leq 2a + 2$ if $a < b$ and $l \leq 2a$ if $a = b$, with only one non-trivial component H . If m_1 and m_2 are even integers such that $m_1, m_2 \geq 4$ and either*

- H is a chain which has a decomposition into an m_1 -path and an m_2 -path; or
- H is a ring which has a decomposition into an m_1 -path and an m_2 -path;

then there exists an $(M, m_1, m_2)^*$ -decomposition of $K_{a,b}$.

Proof Let \mathcal{P} be an $(M)^*$ -packing of $K_{a,b}$ which satisfies the hypotheses of the lemma and let L be its leave. We proceed by induction. First we show that the lemma is true when H is a 2-chain or a 2-ring.

If H is a 2-chain then \mathcal{P} satisfies the hypotheses of Lemma 2.4 and we are finished, so we may suppose that H is a 2-ring. Since $l \leq 2a + 2$ if $a < b$ and $l \leq 2a$ if $a = b$, there must exist vertices u and v in the same part of $K_{a,b}$ such that $\deg_L(u) = 4$ and $\deg_L(v) = 0$ (to see this, separately consider two cases depending on whether the two vertices of degree 4 in L are in the same part of $K_{a,b}$). Let y be a neighbour in L of u . Let \mathcal{P}^\dagger be the $(M)^*$ -packing of $K_{a,b}$ obtained from \mathcal{P} by performing the (u, v) -switch S with origin y . Regardless of the terminus of S , the only non-trivial component of the leave of \mathcal{P} is a 2-chain and we can apply Lemma 2.4 to complete the proof.

We now show that, for each integer $s \geq 3$, if the lemma holds when H is an $(s - 1)$ -chain or an $(s - 1)$ -ring, then it holds when H is an s -chain or an s -ring. The proof splits into two cases.

Case 1. Suppose that H is an s -chain. Let $P = [x_0, x_1, \dots, x_{m_1}]$ be an m_1 -path in H such that the edges in $E(H) \setminus E(P)$ form an m_2 -path. Since every vertex of degree 4 in H must be a vertex of P which is not an end vertex, it is easy to see that x_0 and x_{m_1} are vertices of degree 2 in H such that x_0 is in one end cycle of H and x_{m_1} is in the other end cycle of H . Note that x_0 and x_{m_1} are in the same part of $K_{a,b}$ since m_1 is even. Let L^\dagger be the leave of the $(M)^*$ -packing of $K_{a,b}$ obtained from \mathcal{P} by performing the (x_0, x_{m_1}) -switch S with origin x_1 . If S does not have terminus x_{m_1-1} , then, by Lemma 2.2, L^\dagger has a decomposition into an m_1 -cycle and an m_2 -cycle, and we are finished (note that $x_1 x_{m_1} \notin E(H)$ since H is an s -chain with $s \geq 3$). Otherwise S has terminus x_{m_1-1} , and it can be seen that the only non-trivial

component H^\dagger of L^\dagger is an $(s-1)$ -ring. Furthermore, if we let P^\dagger be the path with $E(P^\dagger) = (E(P) \setminus \{x_0x_1, x_{m_1-1}x_{m_1}\}) \cup \{x_0x_{m_1-1}, x_1x_{m_1}\}$, then it is clear that P^\dagger is an m_1 -path in H^\dagger such that the edges in $E(H^\dagger) \setminus E(P^\dagger)$ form an m_2 -path. Thus we are finished by the inductive hypothesis.

Case 2. Suppose that H is an s -ring. Let $P = [x_0, x_1, \dots, x_{m_1}]$ be an m_1 -path in H such that the edges in $E(H) \setminus E(P)$ form an m_2 -path. Since every vertex of degree 4 in H must be a vertex of P which is not an end vertex, it is easy to see that x_0 and x_{m_1} are vertices of degree 2 in H and that there is a ring cycle C of H such that $x_0, x_{m_1} \in V(C)$. Let u be a vertex of C such that $\deg_H(u) = 4$ and such that there is at least one other vertex of degree 4 in L in the same part of $K_{a,b}$ as u (such a vertex u must exist since $s \geq 3$ and hence there are at least three vertices of degree 4 in L). Let v be a vertex in the same part of $K_{a,b}$ as u such that $\deg_L(v) = 0$ (such a vertex must exist since $l \leq 2a+2$). Let y and z be the neighbours in C of u . Let D be the ring cycle of H , other than C , such that $u \in V(D)$. Let L^\dagger be the leave of the $(M)^*$ -packing of $K_{a,b}$ obtained from \mathcal{P} by performing the (u, v) -switch S with origin y . If the terminus of S is a neighbour in D of u , then the only non-trivial component H^\dagger of L^\dagger is an $(s-1)$ -ring and it can be seen that H^\dagger has a decomposition into an m_1 -path and an m_2 -path (take an m_1 -path with edge set $(E(P) \setminus (E(C) \cup E(D))) \cup E(P') \cup E(P'')$ where P' and P'' are paths in the ring cycle of H^\dagger which is not a ring cycle of H whose lengths add to $|E(P) \cap (E(C) \cup E(D))|$). Thus we are finished by the inductive hypothesis. Otherwise, the terminus of S is z , the only non-trivial component H^\dagger of L^\dagger is an s -chain, and it can be seen that H^\dagger has a decomposition into an m_1 -path and an m_2 -path (take an m_1 -path with edge set $(E(P) \setminus E(C)) \cup E(P')$ where P' is a path of length $|E(P) \cap E(C)|$ in the cycle of H^\dagger which is not a ring cycle of H). Thus we can proceed as we did in Case 1. \square

Lemma 3.2 *Suppose that there exists an $(M)^*$ -packing of $K_{a,b}$ with a leave L of size l with k non-trivial components such that exactly one vertex of L has degree 4 and every other vertex of L has degree 2 or degree 0. If R is one of the parts of $K_{a,b}$ and*

m_1 and m_2 are integers such that $m_1, m_2 \geq k + 1$ and $m_1 + m_2 = l$, then there exists $(M)^*$ -packing of $K_{a,b}$ with a leave whose only non-trivial component is a chain which has a decomposition into an m_1 -path and an m_2 -path such that if $m_1, m_2 \geq 3$ then at least one end vertex of the paths is in R .

Proof Let \mathcal{P} be an $(M)^*$ -packing of $K_{a,b}$ which satisfies the hypotheses of the lemma. It follows that L has one component which is a 2-chain and that any other non-trivial component of L is a cycle. Let the non-trivial components of L be H and C_1, C_2, \dots, C_{k-1} (just H if $k = 1$), where H is the 2-chain. Let h be the size of H and, if $k \geq 2$, let c_i be the size of C_i for $i \in \{1, 2, \dots, k-1\}$.

For each integer s such that $1 \leq s \leq k$ we claim that, for all integers x_1 and x_2 such that $x_1 \geq s + 1$, $x_2 \geq s + 1$ and $x_1 + x_2 = h + c_1 + c_2 + \dots + c_{s-1}$ ($x_1 + x_2 = h$ if $s = 1$), there exists an $(M)^*$ -packing of $K_{a,b}$ with a leave L'' such that the non-trivial components of L'' are H'' and $C_s, C_{s+1}, \dots, C_{k-1}$ (just H'' if $s = k$), where H'' is a chain which has a decomposition into an x_1 -path and an x_2 -path such that if $x_1, x_2 \geq 3$ then at least one end vertex of the paths is in R . Note that if this claim holds for $s = k$ then we have the required result, so it suffices to prove the claim. We will do this by induction on s .

Suppose first that $s = 1$. For any integers x_1 and x_2 such that $x_1, x_2 \geq 2$ and $x_1 + x_2 = h$, it is easy to see that H has a decomposition into an x_1 -path and an x_2 -path such that if $x_1, x_2 \geq 3$ then at least one end vertex of the paths is in R . Thus our claim is true when $s = 1$.

Now suppose that our claim holds when $s = t$ for some integer t such that $1 \leq t < k$. We will show that it holds when $s = t + 1$. Let x_1 and x_2 be integers such that $x_1, x_2 \geq t + 2$ and $x_1 + x_2 = h + c_1 + c_2 + \dots + c_t$. Suppose without loss of generality that $x_1 \leq x_2$. Then we have that $t + 2 \leq x_1 \leq \frac{1}{2}(h + c_1 + c_2 + \dots + c_t)$. It is easy to see that we can find positive integers p and p^\dagger such that $x_1 = p + p^\dagger$, $t + 1 \leq p \leq \frac{1}{2}(h + c_1 + c_2 + \dots + c_{t-1})$ and $1 \leq p^\dagger \leq \frac{1}{2}c_t$. Furthermore, we can choose these integers such that $p \geq 3$ if $x_1 \geq 4$. Let $q = h + c_1 + c_2 + \dots + c_{t-1} - p$ and note

that $q \geq p$ since $p \leq \frac{1}{2}(h + c_1 + c_2 + \cdots + c_{t-1})$.

Thus, by our inductive hypothesis, there exists an $(M)^*$ -packing \mathcal{P}' of $K_{a,b}$ with a leave L' such that the non-trivial components of L' are H' and $C_t, C_{t+1}, \dots, C_{k-1}$, where H' is an f -chain, for some integer $f \geq 2$, which has a decomposition into a p -path P and a q -path Q such that if $p \geq 3$ then at least one end vertex of P and Q is in R . Let u and w be the end vertices of P and Q , where $w \in R$ if $p \geq 3$.

Let y be the neighbour in P of u , let z be the neighbour in Q of u , and let v be a vertex in C_t which is in the same part of $K_{a,b}$ as u . Let L'' be the leave of the $(M)^*$ -packing of $K_{a,b}$ obtained from \mathcal{P}' by performing the (u, v) -switch S with origin y . Then the non-trivial components of L'' are H'' and $C_{t+1}, C_{t+2}, \dots, C_{k-1}$ (or just H'' if $t = k - 1$) where H'' is an $(f + 1)$ -chain if the terminus of S is z and H'' is an f -chain otherwise (if the terminus of S is not z then it is a neighbour in C_t of v). In either case it can be seen that H'' has a decomposition into an x_1 -path and an x_2 -path such that w is an end vertex of each path (take an x_1 -path with edge set $(E(P) \setminus \{uy\}) \cup \{vy\} \cup E(P^\dagger)$ where P^\dagger is a path of length p^\dagger whose edges are in $E(C_t) \cap E(H'')$). The proof is completed by noting that if $p \geq 3$ then $w \in R$ and that if $p = 2$ then $x_1 = 3$ and at least one end vertex of the paths is in R . \square

Lemma 3.3 *Suppose that there exists an $(M)^*$ -packing of $K_{a,b}$ with a leave L of size l with k non-trivial components such that exactly one vertex of L has degree 4, and every other vertex of L has degree 2 or degree 0. If m_1 and m_2 are even integers such that $m_1, m_2 \geq \max(4, k + 1)$ and $m_1 + m_2 = l$, then there is an $(M, m_1, m_2)^*$ -decomposition of $K_{a,b}$.*

Proof Let \mathcal{P} be an $(M)^*$ -packing of $K_{a,b}$ which satisfies the hypotheses of the lemma. By applying Lemma 3.2 to \mathcal{P} we can obtain an $(M)^*$ -packing of $K_{a,b}$ with a leave whose only non-trivial component is a chain which has a decomposition into an m_1 -path and an m_2 -path. We can then obtain an $(M, m_1, m_2)^*$ -decomposition of $K_{a,b}$ by applying Lemma 3.1. \square

Lemma 3.4 *Suppose that there exists an $(M)^*$ -packing of $K_{a,b}$ with a leave L . If u and v are vertices in the same part of $K_{a,b}$ such that $\deg_L(u) > \deg_L(v)$, then there exists an $(M)^*$ -packing of $K_{a,b}$ with a leave L' such that $\deg_{L'}(u) = \deg_L(u) - 2$, $\deg_{L'}(v) = \deg_L(v) + 2$, and $\deg_{L'}(x) = \deg_L(x)$ for all $x \in V(L) \setminus \{u, v\}$. Furthermore, this L' also satisfies*

- (i) *if $\deg_L(v) = 0$ and u is not a cut vertex of L , then L' has the same number of non-trivial components as L ; and*
- (ii) *if $\deg_L(v) = 0$, then either L' has the same number of non-trivial components as L or L' has one more non-trivial component than L .*

Proof Let \mathcal{P} be an $(M)^*$ -packing of $K_{a,b}$ which satisfies the hypotheses of the lemma. Since L is even, we have that $\deg_L(u) \geq \deg_L(v) + 2$ and thus there exists a (u, v) -switch S whose origin is a neighbour in L of u and whose terminus is another neighbour in L of u . Let \mathcal{P}' be the $(M)^*$ -packing of $K_{a,b}$ obtained from \mathcal{P} by performing S . Using the fact that L is an even graph it can be seen that \mathcal{P}' has the required properties. \square

For an $(M)^*$ -packing \mathcal{P} of a graph G we define

$$d(\mathcal{P}) = \frac{1}{2} \sum_{x \in D} (\deg_L(x) - 2),$$

where L is the leave of \mathcal{P} and D is the set of vertices of L having degree at least 4.

Lemma 3.5 *Suppose that there exists an $(M)^*$ -packing \mathcal{P}_0 of $K_{a,b}$, where $a \leq b$, with a leave L_0 of size l , where $l \leq 2a + 2$ if $a < b$ and $l \leq 2a$ if $a = b$, with k_0 non-trivial components such that L_0 has at least one vertex of degree at least 4. Then there exists an $(M)^*$ -packing of $K_{a,b}$ with a leave L' such that exactly one vertex of L' has degree 4, every other vertex of L' has degree 2 or degree 0, and L' has at most $\min(k_0 + d(\mathcal{P}_0) - 1, \lfloor \frac{l}{4} \rfloor - 1)$ non-trivial components.*

Proof Let $d = d(\mathcal{P}_0)$. Note that, informally, $d - 1$ is the minimum number of applications of Lemma 3.4 that would be required to transform \mathcal{P}_0 into an $(M)^*$ -

packing of $K_{a,b}$ in whose leave one vertex has degree 4 and every other vertex has degree 2 or degree 0.

Create a sequence

$$\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{d-1}$$

of $(M)^*$ -packings of $K_{a,b}$ inductively by letting \mathcal{P}_{i+1} be the $(M)^*$ -packing of $K_{a,b}$ obtained by applying Lemma 3.4 to \mathcal{P}_i , choosing u and v to be vertices in the same part of $K_{a,b}$ such that u is of degree at least 4 in the leave of \mathcal{P}_i and v is an isolated vertex in the leave of \mathcal{P}_i . (To see that such vertices exist, consider two cases depending on whether all the vertices with degree at least 4 in the leave of \mathcal{P}_i are in the same part of $K_{a,b}$, noting that $l \leq 2a + 2$ if $a < b$, that $l \leq 2a$ if $a = b$, and that for each $j \in \{0, 1, \dots, d-2\}$ the leave of \mathcal{P}_j contains at least two vertices of degree at least 4 or at least one vertex of degree at least 6.) For each $i \in \{1, 2, \dots, d-1\}$, let L_i be the leave of \mathcal{P}_i and let k_i be the number of non-trivial components in L_i . Note that exactly one vertex of L_{d-1} has degree 4 and every other vertex of L_{d-1} has degree 2 or degree 0.

It only remains to show that $k_{d-1} \leq \min(k_0 + d - 1, \lfloor \frac{l}{4} \rfloor - 1)$. Since exactly one vertex of L_{d-1} has degree 4 and every other vertex of L_{d-1} has degree 2 or degree 0, it follows that L_{d-1} has a decomposition into $k_{d-1} + 1$ cycles. Since each of these cycles has length at least 4, we have that $l \geq 4(k_{d-1} + 1)$ and so $k_{d-1} \leq \lfloor \frac{l}{4} \rfloor - 1$. Also, by Property (ii) of Lemma 3.4, $k_{i+1} \leq k_i + 1$ for each $i \in \{0, 1, \dots, d-2\}$ and hence $k_{d-1} \leq k_0 + d - 1$. \square

Lemma 3.6 *Suppose there exists an $(M, h, m, m')^*$ -decomposition of $K_{a,b}$, where $a \leq b$. If $m + m' \leq 3h$, $m + m' + h \leq 2a + 2$ if $a < b$, and $m + m' + h \leq 2a$ if $a = b$, then there exists an $(M, h, m + m')^*$ -decomposition of $K_{a,b}$.*

Proof Let \mathcal{D} be an $(M, h, m, m')^*$ -decomposition of $K_{a,b}$ which satisfies the hypotheses of the lemma. Let \mathcal{P} be an $(M)^*$ -packing of $K_{a,b}$ obtained by omitting an m -cycle, an m' -cycle and an h -cycle from \mathcal{D} , let L be the leave of \mathcal{P} , and let k be the number

of non-trivial components of L . The proof now splits into cases depending on the properties of the graph L .

Case 1. Suppose that $k = 3$. Then the non-trivial components of L are an m -cycle, an m' -cycle and an h -cycle. Let u and v be vertices in the same part of $K_{a,b}$ such that u and v are in two distinct cycles of L which have lengths m and m' respectively, and let y be a neighbour in L of u . Let \mathcal{P}^\dagger be the $(M)^*$ -packing of $K_{a,b}$ obtained from \mathcal{P} by performing the (u, v) -switch S with origin y and let L^\dagger be the leave of \mathcal{P}^\dagger . Then L^\dagger has exactly two non-trivial components, one an h -cycle and the other either an $(m + m')$ -cycle or an (m, m') -chain. In the former case we can add the two cycles to \mathcal{P}^\dagger to complete the proof. In the latter case, since $m, m', h \geq 4$, we can apply Lemma 3.3 (setting $m_1 = h$ and $m_2 = m + m'$) to complete the proof.

Case 2. Suppose that $k = 2$, and that the non-trivial components of L are a cycle and a 2-chain. Then, since $m, m', h \geq 4$, we can apply Lemma 3.3 (setting $m_1 = h$ and $m_2 = m + m'$) to complete the proof.

Case 3. Suppose that we are in neither Case 1 nor Case 2. Then L contains at least two vertices of degree at least 4 or at least one vertex of degree at least 6. Let $d = d(\mathcal{P})$.

We claim that it suffices to find an $(M)^*$ -packing \mathcal{P}^\dagger of $K_{a,b}$ such that exactly one vertex in the leave of \mathcal{P}^\dagger has degree 4, every other vertex in the leave of \mathcal{P}^\dagger has degree 2 or degree 0, and the number k^\dagger of non-trivial components in the leave of \mathcal{P}^\dagger obeys $k^\dagger \leq \min(m + m' - 1, \lfloor \frac{m+m'+h}{4} \rfloor - 1)$. If we have such a packing then we can apply Lemma 3.3 (setting $m_1 = h$ and $m_2 = m + m'$) to it to complete the proof, provided that $m + m' \geq k^\dagger + 1$ and $h \geq k^\dagger + 1$ both hold (note that $m, m', h \geq 4$). Since $k^\dagger \leq m + m' - 1$ we have that $m + m' \geq k^\dagger + 1$. Since $k^\dagger \leq \lfloor \frac{m+m'+h}{4} \rfloor - 1$, and since $m + m' \leq 3h$ from the hypotheses of the lemma, we have that $k^\dagger \leq h - 1$ and thus $h \geq k^\dagger + 1$.

So it only remains to find such an $(M)^*$ -packing \mathcal{P}^\dagger . Because L contains an h -cycle, it has at least h vertices of positive degree. Our proof now splits into cases

depending on whether L has exactly h vertices of positive degree or not.

Case 3a. Suppose that L has at least $h + 1$ vertices of positive degree. Let \mathcal{P}^\ddagger be the $(M)^*$ -packing of $K_{a,b}$ obtained by applying Lemma 3.5 to \mathcal{P} , let L^\ddagger be the leave of \mathcal{P}^\ddagger , and let k^\ddagger be the number of non-trivial components in L^\ddagger . Note that exactly one vertex of L^\ddagger has degree 4, every other vertex of L^\ddagger has degree 2 or degree 0, and $k^\ddagger \leq \min(k + d - 1, \lfloor \frac{m+m'+h}{4} \rfloor - 1)$. Since we are not in Case 1, $k \in \{1, 2\}$. If $k = 1$ then, since L has at least $h + 1$ vertices of positive degree and since L is an even graph with total degree $2(m + m' + h)$, it can be seen that $d \leq m + m' - 1$ and hence that $k^\ddagger \leq \min(m + m' - 1, \lfloor \frac{m+m'+h}{4} \rfloor - 1)$. If $k = 2$ then L must have at least $h + 4$ vertices of positive degree (note that $m, m' \geq 4$), and since L is an even graph with total degree $2(m + m' + h)$, it can be seen that $d \leq m + m' - 4$ and hence that $k^\ddagger \leq \min(m + m' - 3, \lfloor \frac{m+m'+h}{4} \rfloor - 1)$.

Case 3b. Suppose that L has exactly h vertices of positive degree. Then, since L contains an h -cycle, it follows that $k = 1$ and that L has no cut vertex. Let \mathcal{P}^\dagger be the result of applying Lemma 3.4 to \mathcal{P} , choosing u and v to be vertices in the same part of $K_{a,b}$ such that u is of degree at least 4 in L and v is an isolated vertex in L . (To see that such vertices exist, consider two cases depending on whether all the vertices with degree at least 4 in L are in the same part of $K_{a,b}$, noting that $l \leq 2a + 2$ if $a < b$, that $l \leq 2a$ if $a = b$, and that L contains at least two vertices of degree at least 4 or at least one vertex of degree at least 6.) Note that $d(\mathcal{P}^\dagger) = d - 1$ and that, by Property (i) of Lemma 3.4, the leave of \mathcal{P}^\dagger has exactly one non-trivial component. Now let \mathcal{P}^\ddagger be the $(M)^*$ -packing of $K_{a,b}$ obtained by applying Lemma 3.5 to \mathcal{P}^\dagger , let L^\ddagger be the leave of \mathcal{P}^\ddagger , and let k^\ddagger be the number of non-trivial components in L^\ddagger . Note that exactly one vertex of L^\ddagger has degree 4, every other vertex of L^\ddagger has degree 2 or degree 0, and $k^\ddagger \leq \min(d - 1, \lfloor \frac{m+m'+h}{4} \rfloor - 1)$. Since L has exactly h vertices of positive degree, and since L is an even graph with total degree $2(m + m' + h)$, it can be seen that $d = m + m'$ and hence that $k^\ddagger \leq \min(m + m' - 1, \lfloor \frac{m+m'+h}{4} \rfloor - 1)$. \square

4 Decompositions into short cycles

In this section we will prove Theorem 1.1. Our main tool for this task will be Lemma 3.6. We will also require the following special case of the main result of [13].

Theorem 4.1 ([13]) *Let a and b be integers such that $a, b \geq 4$ and either both a and b are even or $a = b$. Let m_1, m_2, \dots, m_t be integers such that $m_1 + m_2 + \dots + m_t = ab$ if a and b are even, $m_1 + m_2 + \dots + m_t = a^2 - a$ if $a = b$ and a is odd, and $m_i \in \{4, 6\}$ for all $i \in \{1, 2, \dots, t\}$. Then there exists an $(m_1, m_2, \dots, m_t)^*$ -decomposition of $K_{a,b}$.*

Proof of Theorem 1.1 We may assume without loss of generality that $a \leq b$. We claim that there exists an $(M)^*$ -decomposition of $K_{a,b}$ for any list of even integers $M = m_1, m_2, \dots, m_t$ which satisfies $m_1 + m_2 + \dots + m_t = a^2 - a$ if a is odd (and hence $a = b$), $m_1 + m_2 + \dots + m_t = ab$ if a and b are even, and $4 \leq m_1 \leq m_2 \leq \dots \leq m_t \leq \min(a, b, 3m_{t-1})$. Proving this claim will prove the theorem.

If $a \leq 3$ then the claim is vacuous, so we may assume that $a \geq 4$. Suppose for a contradiction that the claim is false. Let $Z = z_1, z_2, \dots, z_t$ be a non-decreasing list which satisfies the conditions of the claim but for which there does not exist a $(Z)^*$ -decomposition of $K_{a,b}$. Further suppose that of all such lists Z is one with a maximum number of entries.

Since there does not exist a $(Z)^*$ -decomposition of $K_{a,b}$, it follows from Theorem 4.1 that $z_t \geq 8$. Let Z^\dagger be the list $z_1, z_2, \dots, z_{t-1}, 4, z_t - 4$, reordered so as to be non-decreasing. Since Z satisfies the conditions of the claim, it is routine to check that Z^\dagger also does. Thus, since Z^\dagger has more entries than Z , there exists a $(Z^\dagger)^*$ -decomposition \mathcal{D}^\dagger of $K_{a,b}$ by our definition of Z . Now, provided that $z_t \leq 3z_{t-1}$ and $z_{t-1} + z_t \leq 2a$, we can apply Lemma 3.6 to \mathcal{D}^\dagger (choosing $m = 4$, $m' = z_t - 4$ and $h = z_{t-1}$) to show that a $(Z)^*$ -decomposition of $K_{a,b}$ exists and hence obtain a contradiction. Since Z satisfies the conditions of the claim, we have that $z_t \leq \min(a, 3z_{t-1})$ and hence that $z_t \leq 3z_{t-1}$ and $z_{t-1} + z_t \leq 2z_t \leq 2a$. This completes the proof. \square

5 Incomplete even-cycle systems

In this section we prove Theorem 1.2. If M is a list with every entry equal to some integer m , then we call an (M) -packing of a graph G an *m -cycle packing of G* and we call an (M) -decomposition of a graph G an *m -cycle decomposition of G* . Before we prove Theorem 1.2 we require two lemmas which will also prove useful in the next section.

Lemma 5.1 *Let $m \geq 4$ be an even integer, let G be a complete bipartite graph each of whose parts has even size at least $m+2$, let R be a part of G , and let l be an integer in $\{4, 6, 8, \dots, 2m-4\}$ such that $|E(G)| \equiv l \pmod{m}$. If p and q are positive even integers such that $p, q \geq l-m$ and $p+q=l$, then there is an m -cycle packing of G whose leave has a decomposition into a p -path and a q -path such that both end vertices of the paths are in R .*

Proof If $l \leq m+2$ then by Theorem 1.1 there is an m -cycle packing of G with a leave whose only non-trivial component is an l -cycle, and it is easy to see that this l -cycle decomposes into the required paths. Thus we may suppose that $l \geq m+4$.

By Theorem 1.1 there is an m -cycle packing of G with a leave whose only non-trivial component is an $(l-m)$ -cycle, and it is easy to see that there is an m -cycle in the packing which shares at least one vertex with the $(l-m)$ -cycle in the leave. Removing this m -cycle from the packing results in an m -cycle packing of G with a leave L of size l such that L has exactly one non-trivial component, L has at least one and at most $l-m$ vertices of degree 4, and every other vertex of L has degree 2 or degree 0. Furthermore, if L has exactly $l-m$ vertices of degree 4 then L has no cut vertex, since in this case the non-trivial component of L has exactly m vertices and contains an m -cycle. So it can be seen that, by applying Lemma 3.5 to this packing, we can obtain an m -cycle packing of G with a leave L' of size l such that exactly one vertex of L' has degree 4, every other vertex of L' has degree 2 or degree 0, and L' has at most $l-m-1$ non-trivial components. Then by applying Lemma 3.2 we

can obtain an m -cycle packing of G whose leave has a decomposition into a p -path and a q -path such that the end vertices of the paths are in R , as required (note that $p, q \geq l - m \geq 4$). \square

Lemma 5.2 *Let A be a set, let S and T be subsets of A and let s' and t' be positive integers. Then there exist subsets S' and T' of A such that $|S'| = s'$, $|T'| = t'$, $S \cap S' = \emptyset$, $T \cap T' = \emptyset$ and $S' \cap T' = \emptyset$ if and only if*

$$(i) \quad |S \cap T| + s' + t' \leq |A|;$$

$$(ii) \quad |S| + s' \leq |A|;$$

$$(iii) \quad |T| + t' \leq |A|.$$

Proof To see that Conditions (i), (ii) and (iii) are necessary for the existence of S' and T' , note that $S \cap T$, S' and T' are pairwise disjoint subsets of A , that S and S' are disjoint subsets of A , and that T and T' are disjoint subsets of A . We will show that these conditions are also sufficient for the existence of the sets.

If $s' \leq |T \setminus S|$, then there exists a set S' such that $|S'| = s'$ and $S' \subseteq T \setminus S$. Since $|T| + t' \leq |A|$, there exists a set T' such that $|T'| = t'$ and $T' \subseteq A \setminus T$. Since $S' \subseteq T \setminus S$ it follows that $S' \cap T' = \emptyset$ and thus S' and T' have the required properties.

If $s' > |T \setminus S|$, then, since $|S| + s' \leq |A|$, there exists a set S' such that $|S'| = s'$, $S' \subseteq A \setminus S$ and $T \setminus S \subseteq S'$. Then, since $|S \cap T| + s' + t' \leq |A|$, there is a set T' such that $|T'| = t'$ and $T' \subseteq A \setminus ((S \cap T) \cup S')$. Since $T \setminus S \subseteq S'$ it follows that $T \cap T' = \emptyset$ and thus S' and T' have the required properties. \square

If A , S and T are sets and s' and t' are integers such that $S, T \subseteq A$ and Conditions (i), (ii) and (iii) of Lemma 5.2 hold, then we will say that the triple (A, S, T) is (s', t') -good.

Proof of Theorem 1.2. Let U and W be disjoint sets of sizes u and $v - u$ respectively. Let ∞ be an element of U . Let G_1 be the complete graph on vertex set $W \cup \{\infty\}$

and let G_2 be the complete bipartite graph with parts $U \setminus \{\infty\}$ and W . Note that $G_1 \cup G_2$ is isomorphic to $K_v - K_u$ and hence that $|E(G_1)| + |E(G_2)| = \binom{v}{2} - \binom{u}{2}$.

Suppose first that $v - u = 4$. Then we have that $m = 4$ (since $4 \leq m \leq v - u$) and hence, from the hypotheses of the theorem, that $\binom{v}{2} - \binom{u}{2} \equiv 0 \pmod{4}$. Also, we have that $|E(G_1)| = \binom{v-u+1}{2} = 10$, that $|E(G_2)| \equiv 0 \pmod{4}$ (since $|E(G_2)| = (u-1)(v-u)$), and hence that $\binom{v}{2} - \binom{u}{2} \equiv 2 \pmod{4}$ (since $|E(G_1)| + |E(G_2)| = \binom{v}{2} - \binom{u}{2}$). This is a contradiction and hence we may assume that $v - u \geq 6$.

Let e be the unique element of $\{0, 4, 6, 8, \dots, m-2, m+2\}$ such that $|E(G_1)| \equiv e \pmod{m}$. It follows from Lemma 2.5 of [15] (noting that $v - u + 1 \geq 7$) that there is an m -cycle packing \mathcal{P}_1 of G_1 whose leave has an e -cycle as its only non-trivial component if $e \neq 0$ and whose leave is empty if $e = 0$. To complete the proof we will find an m -cycle packing of G_2 with a leave L_2 such that the union of L_2 with the leave of \mathcal{P}_1 (after a relabelling of the vertices in \mathcal{P}_1) has a decomposition into m -cycles. Since $|E(G_1)| + |E(G_2)| = \binom{v}{2} - \binom{u}{2}$ and $\binom{v}{2} - \binom{u}{2} \equiv 0 \pmod{m}$, we have that $|E(G_2)| \equiv -e \pmod{m}$. The proof now splits into cases depending on the value of e .

Case 1. Suppose that $e = 0$. Then $|E(G_2)| \equiv 0 \pmod{m}$, by Theorem 1.1 there is an m -cycle decomposition \mathcal{P}_2 of G_2 , and $\mathcal{P}_1 \cup \mathcal{P}_2$ is the required decomposition.

Case 2. Suppose that $4 \leq e \leq m-2$. Since $e \neq 0$ it must be that $u \geq m+3$ and $v - u \geq m+2$ (for otherwise either $u-1 = m$ or $v-u = m$ and in either case we have that $|E(G_2)| \equiv 0 \pmod{m}$ and $e = 0$). By Lemma 5.1 there is an m -cycle packing \mathcal{P}_2 of G_2 whose leave has a decomposition into an $(m-2)$ -path P_2 and an $(m-e+2)$ -path Q_2 such that the end vertices of P_2 and Q_2 are in W (note that $|E(G_2)| \equiv 2m-e \pmod{m}$, that $m-2 \geq m-e$ and that $m-e+2 \geq m-e$). Let y and z be the end vertices of these paths.

We claim that we can relabel the vertices in \mathcal{P}_1 in such a way that its leave has a decomposition into a 2-path P_1 from y to z and an $(e-2)$ -path Q_1 from y to z such that $V(P_1) \cap V(P_2) = \{y, z\}$ and $V(Q_1) \cap V(Q_2) = \{y, z\}$. To see that we can do this, note

that $(W, V(P_2) \cap W, V(Q_2) \cap W)$ is $(1, e-3)$ -good (since $|W| \geq m+2$, $|V(P_2) \cap W| = \frac{m}{2}$, $|V(Q_2) \cap W| = \frac{m-e+4}{2}$, and hence $|(V(P_2) \cap W) \cap (V(Q_2) \cap W)| \leq \frac{m-e+4}{2}$) and use Lemma 5.2. Then $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \{P_1 \cup P_2, Q_1 \cup Q_2\}$ is the required decomposition.

Case 3. Suppose that $e = m + 2$. Again, since $e \neq 0$ it must be that $u \geq m + 3$ and $v - u \geq m + 2$. By Lemma 5.1 there is an m -cycle packing \mathcal{P}_2 of G_2 whose leave has a decomposition into an $(m-4)$ -path P_2 and a 2-path Q_2 such that the end vertices of the paths are in W (note that $|E(G_2)| \equiv m-2 \pmod{m}$, that $m-4 \geq -2$ and that $2 \geq -2$). Let y and z be the end vertices of these paths.

We claim that we can relabel the vertices in \mathcal{P}_1 in such a way that its leave has a decomposition into a 4-path P_1 from y to z and an $(m-2)$ -path Q_1 from y to z such that $V(P_1) \cap V(P_2) = \{y, z\}$ and $V(Q_1) \cap V(Q_2) = \{y, z\}$. To see that we can do this, note that $(W, V(P_2) \cap W, V(Q_2) \cap W)$ is $(3, m-3)$ -good (since $|W| \geq m+2$, $|V(P_2) \cap W| = \frac{m-2}{2}$, $|V(Q_2) \cap W| = 2$, and hence $|(V(P_2) \cap W) \cap (V(Q_2) \cap W)| \leq 2$) and use Lemma 5.2. Then $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \{P_1 \cup P_2, Q_1 \cup Q_2\}$ is the required decomposition.

□

6 Decompositions of Complete Multipartite Graphs

In this section we will use Theorem 1.1 to address the problem of decomposing a complete multipartite graph into even cycles. In particular we will prove Theorem 1.3.

Lemma 6.1 *Let H_2 be a complete bipartite graph with parts A and B of even size, and let H_1 be an even graph with vertex set A . If there exists an (M_1) -packing of H_1 whose leave has a decomposition into a p -path P and a q -path Q , and there exists an (M_2) -packing of H_2 with a leave whose only non-trivial component is an l -cycle, then the following hold.*

- (a) *For any even integers p' and q' such that $p', q' \geq 2$ and $p' + q' = l$, if $(A, V(P), V(Q))$*

is $(\frac{p'-2}{2}, \frac{q'-2}{2})$ -good, then there exists an $(M_1, M_2, p + p', q + q')$ -decomposition of $H_1 \cup H_2$.

(b) For any even integers p' and q' such that $p', q' \geq 2$ and $p' + q' = l$, if $(A, V(P), V(Q))$ is $(\frac{p'-2}{2}, \frac{q'-2}{2})$ -good, then there exists an (M_1, M_2) -packing of $H_1 \cup H_2$ whose leave has a decomposition into a $(p + p')$ -path and a $(q + q')$ -path.

(c) For any even integers p' , q' and p'' such that $p' \geq 2$, $q' \geq 0$, $p'' \geq 2$ and $p' + q' + p'' = l$, if $(A, V(P), V(Q))$ is $(\frac{p'-2}{2}, \max(\frac{q'-2}{2}, 0))$ -good, then there exists an $(M_1, M_2, p + p')$ -packing of $H_1 \cup H_2$ whose leave has a decomposition into a p'' -path and a $(q + q')$ -path.

Proof Let \mathcal{P}_1 be an (M_1) -packing of H_1 whose leave has a decomposition into a p -path P and a q -path Q , and let \mathcal{P}_2 be an (M_2) -packing of H_2 with a leave whose only non-trivial component is an l -cycle. Since H_1 is an even graph, the leave of the \mathcal{P}_1 is an even graph and there are two vertices a and a^\dagger in A such that a and a^\dagger are the end vertices of P and Q . We will show separately that each of (a), (b) and (c) hold.

(a). Using Lemma 5.2, since $(A, V(P), V(Q))$ is $(\frac{p'-2}{2}, \frac{q'-2}{2})$ -good, we can relabel vertices in \mathcal{P}_2 in such a way that the leave of \mathcal{P}_2 has a decomposition into a p' -path P' from a to a^\dagger and a q' -path Q' from a to a^\dagger such that $V(P') \cap V(P) = V(Q') \cap V(Q) = \{a, a^\dagger\}$. Then $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \{P \cup P', Q \cup Q'\}$ is an $(M_1, M_2, p + p', q + q')$ -decomposition of $H_1 \cup H_2$.

(b). Let b and b^\dagger be distinct vertices in B . Using Lemma 5.2, since $(A, V(P), V(Q))$ is $(\frac{p'-2}{2}, \frac{q'-2}{2})$ -good, we can relabel vertices in \mathcal{P}_2 in such a way that the leave of \mathcal{P}_2 has a decomposition into the 1-paths $[a, b]$ and $[a, b^\dagger]$, a $(p' - 1)$ -path P' from a^\dagger to b^\dagger and a $(q' - 1)$ -path Q' from a^\dagger to b such that $V(P') \cap V(P) = V(Q') \cap V(Q) = \{a^\dagger\}$. Then $\mathcal{P}_1 \cup \mathcal{P}_2$ is an (M_1, M_2) -packing of $H_1 \cup H_2$, and $P \cup P' \cup [a, b]$ and $Q \cup Q' \cup [a, b^\dagger]$ are the required paths in its leave.

(c). If $q' = 0$, then using Lemma 5.2, since $(A, V(P), V(Q))$ is $(\frac{p'-2}{2}, 0)$ -good, we can relabel vertices in \mathcal{P}_2 in such a way that the leave of \mathcal{P}_2 has a decomposition into a p' -path P' from a to a^\dagger and a p'' -path P'' from a to a^\dagger such that $V(P') \cap V(P) = \{a, a^\dagger\}$. Then $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \{P \cup P'\}$ is an $(M_1, M_2, p + p')$ -packing of $H_1 \cup H_2$ and P'' and Q are the required paths in its leave.

If $q' \geq 2$, then let b and b^\dagger be distinct vertices in B . Using Lemma 5.2, since $(A, V(P), V(Q))$ is $(\frac{p'-2}{2}, \frac{q'-2}{2})$ -good, we can relabel vertices in \mathcal{P}_2 in such a way that the leave of \mathcal{P}_2 has a decomposition into the 1-path $[a, b^\dagger]$, a p' -path P' from a to a^\dagger , a $(q' - 1)$ -path Q' from a^\dagger to b , and a p'' -path P'' from b to b^\dagger such that $V(P') \cap V(P) = \{a, a^\dagger\}$ and $V(Q') \cap V(Q) = \{a^\dagger\}$. Then $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \{P \cup P'\}$ is an $(M_1, M_2, p + p')$ -packing of $H_1 \cup H_2$, and P'' and $Q \cup Q' \cup [a, b^\dagger]$ are the required paths in its leave.

□

By combining Theorem 1.1 and Lemma 6.1, many decompositions of complete multipartite graphs into even length cycles can be created. Here, we will restrict our focus to decompositions of complete multipartite graphs into uniform even length cycles and content ourselves with proving Theorem 1.3. The proof of this result is indicative of how Theorem 1.1 and Lemma 6.1 can be combined effectively.

Proof of Theorem 1.3. If G has only one part then it has no edges and the result is trivial, so we may assume that G has at least two parts. In [11] necessary and sufficient conditions are found for the existence of a decomposition of a complete multipartite graph into m -cycles in the cases $m = 4$ and $m = 6$. It follows from these conditions that the theorem is true for $m \in \{4, 6\}$, so we may assume that $m \geq 8$. Let the parts of G be V_1, V_2, \dots, V_t . For each $i \in \{1, 2, \dots, t - 1\}$ let G_i be the complete bipartite graph with parts $V_1 \cup V_2 \cup \dots \cup V_i$ and V_{i+1} , and note that G is the edge disjoint union of the graphs G_1, G_2, \dots, G_{t-1} . Also, for each $i \in \{1, 2, \dots, t - 1\}$, let l_i be the element of $\{0, 4, 6, 8, \dots, m - 2, m + 2\}$ such that $|E(G_i)| \equiv l_i \pmod{m}$.

Roughly speaking, we will find the required m -cycle decomposition of G inductively, by beginning with an m -cycle packing of G_1 (obtained using Theorem 1.1)

and, at the i th stage, combining the m -cycle packing of $G_1 \cup G_2 \cup \dots \cup G_i$ which we have already created with an m -cycle packing of G_{i+1} whose leave has size l_{i+1} (this packing of G_{i+1} is also obtained using Theorem 1.1). The packings are combined by either simply taking their union or by applying Lemma 6.1.

Let $\sigma_i = l_{i+1} + l_{i+2} + \dots + l_{t-1}$ for each $i \in \{1, 2, \dots, t-2\}$, and let $\sigma_{t-1} = 0$. Intuitively, σ_i is the sum of the sizes of the leaves of the packings which we are yet to add, when we have just constructed our packing of $G_1 \cup G_2 \cup \dots \cup G_i$.

We claim that for each $s \in \{1, 2, \dots, t-1\}$ there is an m -cycle packing \mathcal{P}_s of $G_1 \cup G_2 \cup \dots \cup G_s$ with a leave L_s such that

- (i) if $\sigma_s \equiv 0 \pmod{2m}$, then L_s is empty;
- (ii) if $\sigma_s \equiv 2 \pmod{2m}$, then L_s has a decomposition into an $(m-4)$ -path and a 2-path;
- (iii) if $\sigma_s \equiv 2m-2 \pmod{2m}$, then L_s has a decomposition into an $(m-2)$ -path and a 4-path;
- (iv) if $\sigma_s \equiv x \pmod{2m}$ for some $x \in \{4, 6, 8, \dots, 2m-4\}$, then L_s has a decomposition into a d_s -path and an e_s -path, where d_s and e_s are the even integers such that $d_s + e_s = 2m - x$ and $d_s \in \{e_s, e_s + 2\}$.

Since $\sigma_{t-1} = 0$, when $s = t-1$ this claim is the result of the theorem, and hence it suffices to prove the claim. We shall do so by induction on s . Note that, since $|E(G)| \equiv 0 \pmod{m}$, we have that $l_1 + l_2 + \dots + l_{t-1} \equiv 0 \pmod{m}$, and hence that $l_1 + l_2 + \dots + l_i + \sigma_i \equiv 0 \pmod{m}$ for each $i \in \{1, 2, \dots, t-1\}$.

Suppose that $s = 1$. Note that, since $l_1 + \sigma_1 \equiv 0 \pmod{m}$, it follows from the definition of l_1 that $|E(G_1)| \equiv -\sigma_1 \pmod{m}$. If $\sigma_1 \equiv 0 \pmod{2m}$, then $|E(G_1)| \equiv 0 \pmod{m}$ and by Theorem 1.1 there is an m -cycle decomposition of G_1 . If $\sigma_1 \equiv 2 \pmod{2m}$, then $|E(G_1)| \equiv m-2 \pmod{m}$ and by Theorem 1.1 there is an m -cycle packing of G_1 with a leave whose only non-trivial component is an $(m-2)$ -cycle.

This $(m - 2)$ -cycle can easily be decomposed into an $(m - 4)$ -path and a 2-path. If $\sigma_1 \equiv 2m - 2 \pmod{2m}$, then $|E(G_1)| \equiv 2 \pmod{m}$ and by Theorem 1.1 there is an m -cycle packing of G_1 with a leave whose only non-trivial component is an $(m + 2)$ -cycle. This $(m + 2)$ -cycle can easily be decomposed into an $(m - 2)$ -path and a 4-path. Finally, if $\sigma_1 \equiv x \pmod{2m}$ for some $x \in \{4, 6, 8, \dots, 2m - 4\}$, then $|E(G_1)| \equiv -x \pmod{m}$ and by Lemma 5.1 (setting $l = 2m - x$) it can be seen that there is an m -cycle packing of G_1 whose leave has a decomposition into a d_1 -path and an e_1 -path (it is easy to see that $d_1, e_1 \geq m - x$ from their definition). So the claim is indeed true in the case $s = 1$.

Now suppose that the claim is true for $s = k$ for some integer k such that $1 \leq k \leq t - 2$, so there is an m -cycle packing \mathcal{P}_k of $G_1 \cup G_2 \cup \dots \cup G_k$ with a leave L_k satisfying (i), (ii), (iii) and (iv) for $s = k$. We will show that the claim is true for $s = k + 1$.

Note that by Theorem 1.1 there is an m -cycle packing \mathcal{C} of G_{k+1} whose leave has an l_{k+1} -cycle as its only non-trivial component if $l_{k+1} \neq 0$ and whose leave is empty if $l_{k+1} = 0$. If $l_{k+1} \geq 8$ then, also by Theorem 1.1, for each $z \in \{4, 6, 8, \dots, l_{k+1} - 4\}$ there exists a packing \mathcal{C}^z of G_k with some number of m -cycles and one $(l_{k+1} - z)$ -cycle with a leave whose only non-trivial component is a z -cycle. By combining \mathcal{P}_k with one of these packings, either by simply taking a union or by applying Lemma 6.1 (setting $H_2 = G_{k+1}$, $A = V_1 \cup V_2 \cup \dots \cup V_{k+1}$, $B = V_{k+2}$ and $H_1 = G_1 \cup G_2 \cup \dots \cup G_k$), we can obtain an m -cycle packing \mathcal{P}_{k+1} of $G_1 \cup G_2 \cup \dots \cup G_{k+1}$ satisfying (i), (ii), (iii) and (iv) for $s = k + 1$. The following list gives an outline of how to accomplish this. Note that, from our definition of $\sigma_1, \sigma_2, \dots, \sigma_t$ we have that $\sigma_{k+1} = \sigma_k - l_{k+1}$. Also note that in each case below where we apply Lemma 6.1 its hypotheses are satisfied since $l_{k+1} \leq m + 2$, the two paths in the leave of \mathcal{P}_k each have length at most $m - 2$, and $|V_1 \cup V_2 \cup \dots \cup V_{k+1}| \geq (k + 1)(m + 2) \geq 2(m + 2)$.

- If $l_{k+1} = 0$ then $\sigma_{k+1} = \sigma_k$ and we take the union of \mathcal{P}_k and \mathcal{C} .
- If $\sigma_k \equiv 0 \pmod{2m}$ and $l_{k+1} \in \{4, 6, 8, \dots, m - 2, m + 2\}$, then $\sigma_{k+1} \equiv y \pmod{2m}$ for some $y \in \{4, 6, 8, \dots, 2m - 4\}$ (note that $m \geq 8$). We take

the union of \mathcal{P}_k and \mathcal{C} and note that the l_{k+1} -cycle in the leave of \mathcal{C} has a decomposition into a d_{k+1} -path and an e_{k+1} -path.

- If $\sigma_k \equiv 2 \pmod{2m}$ and $l_{k+1} = 4$, then $\sigma_{k+1} \equiv 2m - 2 \pmod{2m}$. We apply Lemma 6.1 (b) (setting $p = m - 4$, $p' = 2$, $q = 2$ and $q' = 2$) to \mathcal{P}_k and \mathcal{C} .
- If $\sigma_k \equiv 2 \pmod{2m}$ and $l_{k+1} \in \{6, 8, 10, \dots, m - 2, m + 2\}$, then $\sigma_{k+1} \equiv y \pmod{2m}$ for some $y \in \{4, 6, 8, \dots, 2m - 4\}$ (note that $m \geq 8$). We apply Lemma 6.1 (c) (setting $p = m - 4$, $p' = 4$, $q = 2$, $q' = d_{k+1} - 2$ and $p'' = e_{k+1}$) to \mathcal{P}_k and \mathcal{C} .
- If $\sigma_k \equiv 2m - 2 \pmod{2m}$ and $l_{k+1} \in \{4, 6, 8, \dots, m - 2, m + 2\}$, then $\sigma_{k+1} \equiv y \pmod{2m}$ for some $y \in \{4, 6, 8, \dots, 2m - 4\}$ (note that $m \geq 8$). Note that, in fact, $y \leq 2m - 6$ and hence $d_{k+1} \geq 4$. We apply Lemma 6.1 (c) (setting $p = m - 2$, $p' = 2$, $q = 4$, $q' = d_{k+1} - 4$ and $p'' = e_{k+1}$) to \mathcal{P}_k and \mathcal{C} .
- If $\sigma_k \equiv x \pmod{2m}$ for some $x \in \{4, 6, 8, \dots, 2m - 4\}$ and $4 \leq l_{k+1} \leq x - 4$, then $\sigma_{k+1} \equiv y \pmod{2m}$ for some $y \in \{4, 6, 8, \dots, 2m - 4\}$. Note that, in fact, $y \leq x - 4$ and hence $d_{k+1} \geq d_k + 2$ and $e_{k+1} \geq e_k + 2$. We apply Lemma 6.1 (b) (setting $p = d_k$, $p' = d_{k+1} - d_k$, $q = e_k$ and $q' = e_{k+1} - e_k$) to \mathcal{P}_k and \mathcal{C} .
- If $\sigma_k \equiv x \pmod{2m}$ for some $x \in \{4, 6, 8, \dots, 2m - 4\}$ and $l_{k+1} = x - 2$, then $\sigma_{k+1} \equiv 2 \pmod{2m}$. Note that, since $l_{k+1} = x - 2$ and $l_{k+1} \geq 4$, we have that $x \geq 6$ and hence $e_k \leq m - 4$. We apply Lemma 6.1 (c) (setting $p = d_k$, $p' = m - d_k$, $q = e_k$, $q' = m - 4 - e_k$ and $p'' = 2$) to \mathcal{P}_k and \mathcal{C} .
- If $\sigma_k \equiv x \pmod{2m}$ for some $x \in \{4, 6, 8, \dots, 2m - 4\}$ and $l_{k+1} = x$, then $\sigma_{k+1} \equiv 0 \pmod{2m}$. We apply Lemma 6.1 (a) (setting $p = d_k$, $p' = m - d_k$, $q = e_k$ and $q' = m - e_k$) to \mathcal{P}_k and \mathcal{C} .
- If $\sigma_k \equiv x \pmod{2m}$ for some $x \in \{4, 6, 8, \dots, 2m - 4\}$ and $l_{k+1} = x + 2$, then $\sigma_{k+1} \equiv 2m - 2 \pmod{2m}$. We apply Lemma 6.1 (c) (setting $p = d_k$, $p' = m - d_k$, $q = e_k$, $q' = m - 2 - e_k$ and $p'' = 4$) to \mathcal{P}_k and \mathcal{C} .

- If $\sigma_k \equiv x \pmod{2m}$ for some $x \in \{4, 6, 8, \dots, 2m-4\}$ and $x+4 \leq l_{k+1} \leq m+2$, then $\sigma_{k+1} \equiv y \pmod{2m}$ for some $y \in \{4, 6, 8, \dots, 2m-4\}$ (since $m \geq 8$). Note that, in fact, $y = 2m - (l_{k+1} - x)$ and that $l_{k+1} \geq 8$. We apply Lemma 6.1 (a) (setting $p = d_k$, $p' = m - d_k$, $q = e_k$ and $q' = m - e_k$) to \mathcal{P}_k and \mathcal{C}^x , remove the $(l_{k+1} - x)$ -cycle from this packing, and note that this cycle has a decomposition into a d_{k+1} -path and an e_{k+1} -path. \square

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